PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Spatially Periodic Dynamos

G. O. Roberts

Phil. Trans. R. Soc. Lond. A 1970 **266**, 535-558
doi: 10.1098/rsta.1970.0011

SPATIALLY PERIODIC DYNAMOS

By G. O. ROBERTS

Department of Applied Mathematics, University of Newcastle upon Tyne

(*Communicated by Sir Edward Bullard, F.R.S.—Received* 18 *June* 1969)

CONTENTS

It is established analytically that, in a precisely defined sense, almost all steady spatially periodic motions of a homogeneous conducting fluid will give dynamo action at almost all values of the conductivity. The same result is obtained for motions periodic in space-time. The asymptotic form of the growing field, for an arbitrary initial field of finite energy, is also presented.

Dynamo action is first shown to require that for some real vector $\boldsymbol{j}$ there is a magnetic field solution of the form

$$\boldsymbol{B} = \boldsymbol{H} \exp (pt + \mathrm{i}\boldsymbol{j} . \boldsymbol{x}),$$

where $\boldsymbol{H}$ is a complex function of position (or of position and time) with the same periodicity as the motion, and p has positive real part, indicating growth. This number p is an eigenvalue of a linear differential operator on the space of admissible functions $\boldsymbol{H}$. The first term of a power series in $\boldsymbol{j}$ for the eigenvalues p which vanish to zero order is studied. It is thus proved sufficient for dynamo action that the determinant of the symmetric part of a certain 3×3 tensor, a function of the motion and conductivity, is non-zero. Finally, it is shown that this determinant is an analytic function of the conductivity, and is non-zero in a small conductivity limit for nearly all motions. This proves the stated result.

1. Introduction

(*a*) *The kinematic dynamo problem*

The observed terrestrial, stellar and galactic magnetic fields are widely thought to be due to magnetohydrodynamic dynamo action. Much study has therefore been given in recent years to the excitation of a magnetic field by a fluid motion. Because of the difficulty of proving such dynamo action, attention has largely been confined to the equations for the variation of the magnetic field, with an assumed fluid motion, not necessarily steady. In this paper the same kinematic approach is used.

With the usual notation; the equations for the magnetic field in a fluid with uniform conductivity σ e.m.u. are

$$\dot{\boldsymbol{B}} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) + \lambda \nabla^2 \boldsymbol{B}, \tag{1.1}$$

$$\nabla . \boldsymbol{B} = 0, \tag{1.2}$$

where λ is the resistivity or magnetic diffusivity $1/4\pi\sigma$. Equation (1.2) can be regarded as an initial condition, since the divergence of equation (1.1) gives

$$\nabla . \dot{\boldsymbol{B}} = \lambda \nabla^2 (\nabla . \boldsymbol{B}),$$

and with $\nabla . \boldsymbol{B} = 0$ initially, this equation has the unique solution $\nabla . \boldsymbol{B} = 0$ for all time.

The magnetic Reynolds number R, defined by

$$R = UL/\lambda, \tag{1.3}$$

where U and L are respectively velocity and length scales for the motion and are normally chosen as unity, gives an estimate for the relative magnitude of the two terms on the right-hand side of equation (1.1). Dynamo action with a given motion and magnetic Reynolds number requires that with an initial magnetic field of general form it is possible for the total magnetic field energy E, given by

$$E = \int \frac{\boldsymbol{B}^2}{8\pi} \, \mathrm{d}V, \tag{1.4}$$

to grow indefinitely.

(*b*) *Previous work*

P. H. Roberts (1967*a*, *b*) has reviewed previous work on this problem. Cowling (1933, 1957) showed that an axisymmetric magnetic field cannot be amplified or maintained by a fluid motion. The same applies to fields which are functions of only two Cartesian coordinates. The kinematic dynamo problem is thus that of choosing motions complicated enough to give dynamo action and simple enough for this action to be proved.

It appears that all published proofs of dynamo action use, explicitly or implicitly, expansions in ascending powers of small dimensionless parameters, keeping only a few terms. Apart from the magnetic Reynolds number, the only dimensionless numbers available are ratios of different velocity, length and time scales, and suitable combinations of these are used. Thus Backus (1958), with a spherical geometry, proposes 'jerky' motions in which almost instantaneous fluid displacements alternate with long periods when the fluid is stationary; it is clear that an implicit limit is involved, and such motions are far from realistic. Herzenberg (1958) proposes a motion in which small spheres of fluid rotate steadily within a large fluid sphere at rest; his velocity discontinuities are an implicit limit and he also utilizes two explicit limits, the radii of the small spheres being small compared with their separation, and this small compared with the radius of the large sphere. Braginskii (1964*a*, *b*) also takes a spherical geometry, with an axisymmetric toroidal motion defining the large magnetic Reynolds number R, and with poloidal and non-axisymmetric motions smaller by factors of R and $R^{\frac{1}{2}}$ respectively. His expansions for the magnetic field are then assumed to be asymptotically valid as R tends to infinity.

Earlier work by Childress (1967*a*, 1969) and G. O. Roberts (1969, 1970*a*) on spatially periodic dynamos also uses such small parameter approaches, with a limit of small conductivity and with small ratios of length scales. This work is discussed in relation to the present paper in § 6.

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(c) *Spatially periodic dynamos*

In this paper too a small parameter approach is used, but this is done in such a way that generality is preserved in the motion and conductivity. Attention is confined to motions of an infinite fluid with their components either spatially periodic functions or periodic functions of space-time. This has three advantages. First, there is no need to consider boundary conditions, apart from the condition that the total energy given by equation (1.4) is finite. Secondly, Fourier analysis can be readily applied; this is in contrast to the spherical geometry for which the natural decay modes involve spherical harmonics and Bessel functions. Finally, the infinite geometry means that the magnetic field can have an arbitrarily large length scale superposed on the natural length scale of the motion. This gives the small parameter used in the analysis. The natural decay rate of the field is of order the square of this parameter; it is shown that there is a regenerative term of the order of the small parameter itself.

In most of the paper, attention is confined to steady spatially periodic motions; the extension of the analysis to motions periodic in space-time is presented in appendix A. Two linear differential operators and a compact linear operator are studied in the text; required results concerning their spectral properties are presented in appendix B. Appendix C completes the analysis in § 2 of the form of the growing magnetic field; and in appendix D the basic perturbation method of § 3 is justified. At a first reading, therefore, the main text is self-contained.

2. Spatially periodic motions and the corresponding magnetic field form

If there is a fixed set of three independent vectors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $\boldsymbol{l}_3$ such that

$$f(\boldsymbol{x}+\boldsymbol{l}_i) = f(\boldsymbol{x}), \tag{2.1}$$

for all $\boldsymbol{x}$ and for $i = 1$, 2 and 3, then the function $f(\boldsymbol{x})$ will be called spatially periodic. A function with the same periodicity as the motion $\boldsymbol{u}(\boldsymbol{x})$ under consideration will be called $\boldsymbol{u}$-periodic. The average part f^{A} and the oscillatory part $f'(\boldsymbol{x})$ are defined as

$$\begin{aligned} f^{\mathrm{A}} &= \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2 \int_0^1 \mathrm{d}\xi_3\, f(\boldsymbol{l}_1\xi_1+\boldsymbol{l}_2\xi_2+\boldsymbol{l}_3\xi_3) \\ &= \frac{1}{\tau}\int_\tau f(\boldsymbol{x})\,\mathrm{d}V \end{aligned} \tag{2.2}$$

(where τ is the volume defined by the first integral),

$$f'(\boldsymbol{x}) = f(\boldsymbol{x}) - f^{\mathrm{A}}. \tag{2.3}$$

Then $f'(\boldsymbol{x})$ has zero average, and

$$(\nabla f)^{\mathrm{A}} = 0. \tag{2.4}$$

Except in appendix A, attention is confined to motions with

$$\boldsymbol{u}^{\mathrm{A}} = 0. \tag{2.5}$$

The analysis of §§ 3 and 5 can be extended in a straightforward manner to include spatially periodic motions not satisfying condition (2.5), with the same results in that equation (3.11) is unaltered. This extension is not presented, for simplicity, since in any case the analysis of appendix A applies to all such motions.

The $\boldsymbol{u}$-periodic function $f(\boldsymbol{x})$ can be regarded as a function ϕ of the variable $\boldsymbol{\xi}$, where

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{l}_1\xi_1 + \boldsymbol{l}_2\xi_2 + \boldsymbol{l}_3\xi_3 \\ &= L\boldsymbol{\xi}, \end{aligned} \tag{2.6}$$

and the 3×3 matrix L with columns $\boldsymbol{l}_i$ is non-singular. This function $\phi(\boldsymbol{\xi})$ is then periodic in each of ξ_1, ξ_2 and ξ_3, with unit period, and can be written as the Fourier series

$$\phi(\boldsymbol{\xi}) = \sum_{\boldsymbol{n}} \hat{\phi}(\boldsymbol{n}) \exp(2\pi \mathrm{i}\boldsymbol{n}.\boldsymbol{\xi}), \tag{2.7}$$

where the sum is over all sets of three integers. Thus

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in K} \hat{f}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}.\boldsymbol{x}), \tag{2.8}$$

where K is the vector set defined by

$$K = \{2\pi(L^{\mathrm{T}})^{-1}\boldsymbol{n}\colon n_1, n_2 \text{ and } n_3 \text{ integers}\}, \tag{2.9}$$

L^{T} denoting the transpose of L. Further,

$$f^{\mathrm{A}} = \hat{f}(\boldsymbol{0}). \tag{2.10}$$

Any $\boldsymbol{u}$-periodic function, including $\boldsymbol{u}(\boldsymbol{x})$ itself, can be Fourier analysed in this way.

In the basic analysis of § 3, attention is confined to magnetic fields of the form

$$\boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{H}(\boldsymbol{x}) \exp(pt + \mathrm{i}\boldsymbol{j}.\boldsymbol{x}), \tag{2.11}$$

where $\boldsymbol{H}(\boldsymbol{x})$ is complex and $\boldsymbol{u}$-periodic, p is complex, and $\boldsymbol{j}$ is real. A field of this form does not have finite energy; but it is established below that an initial magnetic field of finite energy E given by equation (1.4) can have this energy grow exponentially if, and only if, there is for some real vector $\boldsymbol{j}$ a magnetic field solution of the above form with the real part of p positive. The result is related to Bloch's theorem in solid-state physics, where a partial differential equation with spatially periodic coefficients also arises. With this assumption, it is possible in obtaining the main result to proceed directly to § 3.

Substitution of equation (11) into the magnetic field equations (1.1) and (1.2) gives

$$p\boldsymbol{H}(\boldsymbol{x}) = \mathscr{P}\boldsymbol{H}(\boldsymbol{x}), \tag{2.12}$$

$$(\nabla + \mathrm{i}\boldsymbol{j}).\boldsymbol{H}(\boldsymbol{x}) = 0, \tag{2.13}$$

where $\mathscr{P}$ is a linear differential operator, a function of λ and $\boldsymbol{j}$ as well as of the motion, on the vector space $V(\boldsymbol{j})$ of complex $\boldsymbol{u}$-periodic vector functions satisfying condition (2.13), and

$$\mathscr{P}\boldsymbol{H}(\boldsymbol{x}) = (\nabla + \mathrm{i}\boldsymbol{j}) \times (\boldsymbol{u}\times\boldsymbol{H}) + \lambda(\nabla + \mathrm{i}\boldsymbol{j})^2\boldsymbol{H}. \tag{2.14}$$

Thus p is an eigenvalue of $\mathscr{P}$. A detailed description of the spectral properties of $\mathscr{P}$ is given in appendix B. For the analysis below it is assumed that $\mathscr{P}$ has a set of eigensolutions

$$\{p_n(\boldsymbol{j}),\ \boldsymbol{H}_n(\boldsymbol{x},\boldsymbol{j})\colon n \geqslant 1\}$$

with the eigenvectors complete in the space $V(\boldsymbol{j})$. The more general situation is analysed in appendix C.

Any initial magnetic field $\boldsymbol{B}(\boldsymbol{x})$ with finite total energy can be Fourier analysed in the form

$$\boldsymbol{B}(\boldsymbol{x}) = \int \hat{\boldsymbol{B}}(\boldsymbol{j}) \exp(\mathrm{i}\boldsymbol{j}.\boldsymbol{x})\, \mathrm{d}^3\boldsymbol{j}, \tag{2.15}$$

where, using equation (1.4),

$$\hat{\boldsymbol{B}}(\boldsymbol{j}) = (2\pi)^{-3} \int \boldsymbol{B}(\boldsymbol{x}) \exp(-\mathrm{i}\boldsymbol{j}.\boldsymbol{x})\, \mathrm{d}^3\boldsymbol{x}, \tag{2.16}$$

$$E = \pi^2 \int |\hat{\boldsymbol{B}}(\boldsymbol{j})|^2\, \mathrm{d}^3\boldsymbol{j}. \tag{2.17}$$

It is natural to assume that $\boldsymbol{B}(\boldsymbol{x})$ is infinitely differentiable, since the diffusion term will ensure that this becomes so immediately if it is not so initially. Then, for any n,

$$\hat{\boldsymbol{B}}(\boldsymbol{j}) = o(|\boldsymbol{j}|^{-n}), \tag{2.18}$$

uniformly, as $|\boldsymbol{j}|\rightarrow\infty$. It is also convenient to assume that $\boldsymbol{B}(\boldsymbol{x})$ is smaller than r^{-n} at infinity for any n; this property is permanent according to equation (1.1). Then $\hat{\boldsymbol{B}}(\boldsymbol{j})$ is infinitely differentiable.

The function $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{j})$ is defined by the absolutely convergent sum

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{j}) = \sum_{\boldsymbol{k}\in K} \hat{\boldsymbol{B}}(\boldsymbol{j}+\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x}), \tag{2.19}$$

where the set K is defined by equation (2.9) and the absolute convergence is implied by equation (2.18). Clearly, $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{j})$ is $\boldsymbol{u}$-periodic. Further, it is a differentiable function of $\boldsymbol{j}$, since $\hat{\boldsymbol{B}}(\boldsymbol{j})$ is differentiable, and is in the space $V(\boldsymbol{j})$ defined by equation (2.13), since from equations (2.15) and (1.2),

$$\boldsymbol{j}\,.\,\hat{\boldsymbol{B}}(\boldsymbol{j}) = 0,$$

$$(\nabla+\mathrm{i}\boldsymbol{j})\,.\,\{\hat{\boldsymbol{B}}(\boldsymbol{j}+\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k}\,.\,\boldsymbol{x})\} = 0.$$

Thus, by the assumption of completeness stated above,

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{j}) = \sum_{n=1}^{\infty} a_n(\boldsymbol{j})\, \boldsymbol{H}_n(\boldsymbol{x},\boldsymbol{j}), \tag{2.20}$$

where the $a_n(\boldsymbol{j})$ are complex amplitudes. Now from equations (2.15) and (2.19)

$$\boldsymbol{B}(\boldsymbol{x}) = \int_J \boldsymbol{H}(\boldsymbol{x},\boldsymbol{j}) \exp(\mathrm{i}\boldsymbol{j}\,.\,\boldsymbol{x})\, \mathrm{d}^3\boldsymbol{j}, \tag{2.21}$$

where J is the set of three-dimensional vectors defined by

$$J = \{2\pi(L^{\mathrm{T}})^{-1}\boldsymbol{\nu}\colon\ -\tfrac{1}{2} < \nu_i \leqslant \tfrac{1}{2}\}, \tag{2.22}$$

cf. the definition (2.9) of the vector set K. Thus

$$\boldsymbol{B}(\boldsymbol{x}) = \int_J \left\{\sum_{n=1}^{\infty} a_n(\boldsymbol{j})\, \boldsymbol{H}_n(\boldsymbol{x},\boldsymbol{j})\right\} \exp(\mathrm{i}\boldsymbol{j}\,.\,\boldsymbol{x})\, \mathrm{d}^3\boldsymbol{j}, \tag{2.23}$$

and, using equation (2.11),

$$\boldsymbol{B}(\boldsymbol{x},t) = \int_J \mathrm{d}^3\boldsymbol{j} \sum_{n=1}^{\infty} a_n(\boldsymbol{j})\, \boldsymbol{H}_n(\boldsymbol{x},\boldsymbol{j}) \exp\{p_n(\boldsymbol{j})\, t+\mathrm{i}\boldsymbol{j}\,.\,\boldsymbol{x}\}. \tag{2.24}$$

The result, clearly indicated by equation (2.24), that there is dynamo action if and only if one of the $p_n(\boldsymbol{j})$ has positive real part, is confirmed by the detailed analysis in appendix C. The asymptotic form of the growing field is also derived there, for a general initial field, in the form

$$\boldsymbol{B}(\boldsymbol{x},t) \sim 2\,\mathrm{Re}\,[a_1(\boldsymbol{j}_0)\, \boldsymbol{H}_1(\boldsymbol{x},\boldsymbol{j}_0) \exp\{p_1(\boldsymbol{j}_0)\, t+\mathrm{i}\boldsymbol{j}_0\,.\,\boldsymbol{x}\}\, I(\boldsymbol{x},t)], \quad \text{as} \quad t\rightarrow\infty, \tag{2.25}$$

where $\mathrm{Re}\{p_n(\boldsymbol{j})\}$ has its maximum for $p_1(\pm\boldsymbol{j}_0)$, $a_1(\boldsymbol{j}_0)$ is the corresponding complex amplitude determined by the initial conditions, and the exponential term represents a growing progressive plane wave. The complex amplitude function $I(\boldsymbol{x},t)$ is given by

$$I(\boldsymbol{x},t) = (\pi/t)^{\frac{3}{2}} (\det P)^{-\frac{1}{2}} \exp\{-(\boldsymbol{x}-\boldsymbol{f}t)'\, P^{-1}(\boldsymbol{x}-\boldsymbol{f}t)/4t\}, \tag{2.26}$$

$\boldsymbol{f}$ being the real vector determined by

$$-\mathrm{i}f_k = \frac{\partial p_1}{\partial j_k}(\boldsymbol{j}_0), \tag{2.27}$$

and P the symmetric complex matrix with positive definite real part determined by

$$-P_{km} = \frac{1}{2}\frac{\partial^2 p_1}{\partial j_k \partial j_m}(\boldsymbol{j}_0). \tag{2.28}$$

$I(\boldsymbol{x}, t)$ has its maximum amplitude at $\boldsymbol{x} = \boldsymbol{f}t$ and is small for

$$\mathrm{Re}\{(\boldsymbol{x}-\boldsymbol{f}t)' P^{-1}(\boldsymbol{x}-\boldsymbol{f}t)\} \gg t;$$

further, its space integral is
$$\int I(\boldsymbol{x}, t)\, \mathrm{d}^3\boldsymbol{x} = 8\pi^3. \tag{2.29}$$

Finally, the asymptotic total energy is derived in appendix C in the form

$$E(t) \sim (\tfrac{1}{2}\pi^7)^{\frac{1}{2}} |a_1(\boldsymbol{j}_0)|^2 [\det\{\mathrm{Re}\,(P)\}]^{-\frac{1}{2}} \left\{(1/\tau)\int_\tau |\boldsymbol{H}_1(\boldsymbol{x}, \boldsymbol{j}_0)|^2\, \mathrm{d}\tau\right\} t^{-\frac{3}{2}} \exp[2\,\mathrm{Re}\{p_1(\boldsymbol{j}_0)\}t], \tag{2.30}$$

as $t \to \infty$.

3. Basic analysis

It has been established in § 2 that the motion gives dynamo action if, and only if, there are magnetic field solutions of the form (2.11) with $\mathrm{Re}\,(p) > 0$, for some real $\boldsymbol{j}$. In this section attention is confined to small $\boldsymbol{j}$, and it is assumed that a particular eigenvalue p of the linear differential operator $\mathscr{P}(\boldsymbol{j})$ defined by equation (2.14) can be expanded as

$$p = \sum_0^\infty p_n, \tag{3.1}$$

where p_n is explicitly of order n in the components of $\boldsymbol{j}$. It is further assumed that the corresponding eigenfunction $\boldsymbol{H}(\boldsymbol{x})$, with its arbitrary multiplying constant suitably chosen, can be expanded as

$$\boldsymbol{H}(\boldsymbol{x}) = \sum_0^\infty \boldsymbol{H}_n(\boldsymbol{x}), \tag{3.2}$$

where $\boldsymbol{H}_n(\boldsymbol{x})$ is explicitly of order n in the components of $\boldsymbol{j}$. These assumptions are justified in appendix D, and related to degenerate perturbation theory for linear operators. The approximation in this section of taking only the terms p_0 and p_1 in the expansion of p is shown to be valid for sufficiently small $\boldsymbol{j}$, and it is also shown how further terms in the expansion can in principle be found.

To zero order in the components of $\boldsymbol{j}$, equations (2.12) to (2.14) give

$$p_0 \boldsymbol{H}_0 = \nabla \times (\boldsymbol{u} \times \boldsymbol{H}_0) + \lambda \nabla^2 \boldsymbol{H}_0, \tag{3.3}$$

$$\nabla \cdot \boldsymbol{H}_0 = 0. \tag{3.4}$$

With the use of equation (2.4), the average of equation (3.3) gives

$$p_0 \boldsymbol{H}_0^{\mathrm{A}} = 0. \tag{3.5}$$

This result is the familiar one that the uniform part of a magnetic field cannot grow or decay if the electric field is bounded at infinity. Attention is confined to eigensolutions with

$$\boldsymbol{H}_0^{\mathrm{A}} \neq 0, \quad p_0 = 0. \tag{3.6}$$

To first order in the components of $\boldsymbol{j}$, the average parts of equations (2.12) and (2.13) now give

$$p_1 \boldsymbol{H}_0^{\mathrm{A}} = \mathrm{i}\boldsymbol{j} \times (\boldsymbol{u} \times \boldsymbol{H}_0)^{\mathrm{A}} = \mathrm{i}\boldsymbol{j} \times (\boldsymbol{u} \times \boldsymbol{H}_0')^{\mathrm{A}}, \tag{3.7}$$

using equation (2.5), and
$$\boldsymbol{j} \cdot \boldsymbol{H}_0^{\mathrm{A}} = 0. \tag{3.8}$$

For any given motion, it is established in § 5 that equations (3.3), (3.4) and (3.6) have a unique solution $\boldsymbol{H}_0(\boldsymbol{x})$, for any given $\boldsymbol{H}_0^{\mathrm{A}}$, for all positive values of the resistivity λ, except possibly a bounded discrete set, at most denumerable and with no non-zero point of accumulation. Thus the term $(\boldsymbol{u}\times\boldsymbol{H}_0')^{\mathrm{A}}$ in equation (3.7) is determined by $\boldsymbol{H}_0^{\mathrm{A}}$ in the form

$$\{(\boldsymbol{u}\times\boldsymbol{H}_0')^{\mathrm{A}}\}_q = \alpha_{qr}\{\boldsymbol{H}_0^{\mathrm{A}}\}_r, \tag{3.9}$$

where the nine terms of α_{qr} are real functions of the motion, conductivity and coordinate system, and therefore form a tensor since their contraction with an arbitrary vector gives a vector. With

$$\boldsymbol{b} = \boldsymbol{H}_0^{\mathrm{A}}, \tag{3.10}$$

equations (3.7), (3.8) and (3.9) give

$$p_1 b_q = \mathrm{i}\epsilon_{qrs} j_r \alpha_{sw} b_w, \tag{3.11}$$

$$j_q b_q = 0. \tag{3.12}$$

This three-dimensional eigenvalue problem is analysed in § 4.

4. Analysis of the simplified eigenvalue problem

The three-dimensional eigenvalue problem of equations (3.11) and (3.12) can be written

$$pb_q = \beta_{qw} b_w, \tag{4.1}$$

$$j_q b_q = 0, \tag{4.2}$$

where

$$\beta_{qw} = \mathrm{i}\epsilon_{qrs} j_r \alpha_{sw}. \tag{4.3}$$

Dynamo action at first order only requires that β_{qw} has an eigenvalue β with positive real part, for if p is non-zero, the contraction of equation (4.1) with j_q shows that equation (4.2) is satisfied. Further, β_{qw} is singular, since

$$j_q \beta_{qw} = 0. \tag{4.4}$$

The real tensor α_{sw} can be written as the sum of its symmetric and antisymmetric parts, as follows

$$\alpha_{sw} = \alpha_{\overline{sw}} + \alpha_{\widehat{sw}}, \tag{4.5}$$

where

$$\left.\begin{aligned} \alpha_{\overline{sw}} &= \tfrac{1}{2}(\alpha_{sw}+\alpha_{ws}),\\ \alpha_{\widehat{sw}} &= \tfrac{1}{2}(\alpha_{sw}-\alpha_{ws})\\ &= -\epsilon_{swr} v_r,\\ 2v_r &= -\epsilon_{swr}\alpha_{sw}. \end{aligned}\right\} \tag{4.6}$$

Hence, from equations (4.3), (4.5) and (4.6),

$$\beta_{qw} = \mathrm{i}(\gamma_{qw} - \delta_{qw} j_r v_r + v_q j_w), \tag{4.7}$$

where

$$\gamma_{qw} = \epsilon_{qrs} j_r \alpha_{\overline{sw}}. \tag{4.8}$$

As with β_{qw}, γ_{qw} is singular, and the eigenvector b_w corresponding to any non-zero eigenvalue γ must have $j_w b_w = 0$. Thus from equation (4.7) it is also an eigenvector of β_{qw}, corresponding to the eigenvalue

$$\beta = \mathrm{i}(\gamma - j_r v_r). \tag{4.9}$$

So there is dynamo action to first order if and only if γ_{qw} has an eigenvalue with negative imaginary part.

With axes the principal axes of $\alpha_{s\overline{w}}$,

$$\gamma_{qw} = \begin{bmatrix} 0 & -j_3\alpha_2 & j_2\alpha_3 \\ j_3\alpha_1 & 0 & -j_1\alpha_3 \\ -j_2\alpha_1 & j_1\alpha_3 & 0 \end{bmatrix}, \tag{4.10}$$

where α_1, α_2 and α_3 are the real eigenvalues of $\alpha_{s\overline{w}}$. The non-zero eigenvalues of γ_{qw} are determined by the equation

$$-\gamma^2 = D, \tag{4.11}$$

where

$$D = j_1^2\alpha_2\alpha_3 + j_2^2\alpha_3\alpha_1 + j_3^2\alpha_1\alpha_2$$
$$= j_r j_s \alpha_{rs}^{\dagger} \tag{4.12}$$

the last expression, in which $\alpha_{rs}^{\dagger}$ is the adjoint of $\alpha_{\overline{rs}}$, being invariant. So dynamo action at this order requires just that D be positive.

There are now four possible cases, the first two being the most general:

(*a*) The three α values all have the same sign, $D > 0$ for all $\boldsymbol{j}$, and $\operatorname{Re}(p) > 0$ for sufficiently small $\boldsymbol{j}$ in all directions.

(*b*) One α value has the opposite sign to that of the other two, $D > 0$ only for $\boldsymbol{j}$ directions sufficiently near the corresponding eigenvector of $\alpha_{q\overline{w}}$, and $\operatorname{Re}(p)$ has only been proved positive for sufficiently small $\boldsymbol{j}$ in those directions.

(*c*) One α value is zero and the others have the same sign, $\operatorname{Re}(p)$ has been proved positive for sufficiently small $\boldsymbol{j}$ in all directions not normal to the eigenvector of $\alpha_{r\overline{s}}$ corresponding to the zero eigenvalue.

(*d*) One α value is zero and the others have opposite signs, or more than one is zero, so that $D \leqslant 0$ and there is no dynamo action to first order in the components of $\boldsymbol{j}$.

It may be noted that the determinant $\alpha = |\alpha_{\overline{rs}}|$ is the product of the three eigenvalues; thus if this determinant is non-zero for a particular motion and conductivity, dynamo action occurs. This is the key to the argument of § 5.

Since in the analysis above attention has been confined to small $\boldsymbol{j}$, the field solution in the form (2.11) can be represented as the sum of a part $\boldsymbol{B}^{\mathrm{S}}$, a slowly varying function of position, and $\boldsymbol{B}^{\mathrm{R}}$, a rapidly varying function, where

$$\boldsymbol{B}^{\mathrm{S}} = \boldsymbol{H}^{\mathrm{A}} \exp(pt + \mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}), \tag{4.13}$$

and

$$\boldsymbol{B}^{\mathrm{R}} = \boldsymbol{H}'(\boldsymbol{x}) \exp(pt + \mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}). \tag{4.14}$$

It is of interest to examine the general form of the slowly varying part $\boldsymbol{B}^{\mathrm{S}}$ of the growing fields. The exponential factor indicates that it is a plane wave. Equation (3.8) shows that the magnetic field is normal to $\boldsymbol{j}$, and thus that it is in the wavefronts. With the real and imaginary parts of p and $\boldsymbol{H}^{\mathrm{A}}$ written as p_{r} and p_{i}, $\boldsymbol{H}_{\mathrm{r}}^{\mathrm{A}}$ and $\boldsymbol{H}_{\mathrm{i}}^{\mathrm{A}}$,

$$\operatorname{Re}(\boldsymbol{B}^{\mathrm{S}}) = \{\boldsymbol{H}_{\mathrm{r}}^{\mathrm{A}} \cos(\boldsymbol{j}\cdot\boldsymbol{x} + p_{\mathrm{i}}t) - \boldsymbol{H}_{\mathrm{i}}^{\mathrm{A}} \sin(\boldsymbol{j}\cdot\boldsymbol{x} + p_{\mathrm{i}}t)\}\,\mathrm{e}^{p_{\mathrm{r}}t}. \tag{4.15}$$

Thus unless $\boldsymbol{H}_{\mathrm{r}}^{\mathrm{A}}$ and $\boldsymbol{H}_{\mathrm{i}}^{\mathrm{A}}$ are parallel, the slowly varying part of the magnetic field solution is an elliptically polarized plane wave, of very large wavelength.

This section is concluded with the first order analysis for motions for which the tensor $\alpha_{s\overline{w}}$ is the isotropic tensor

$$\alpha_{s\overline{w}} = \alpha\delta_{sw}. \tag{4.16}$$

With axes so that $\boldsymbol{j} = (j, 0, 0)$,

$$\gamma_{qw} = j\alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \tag{4.17}$$

with the eigensolution

$$\gamma = -\mathrm{i}\alpha j, \quad \boldsymbol{b} = (0, 1, \mathrm{i}). \tag{4.18}$$

Thus, in the first-order approximation,

$$\boldsymbol{B}^{\mathrm{S}} = [0, \cos\{j(x-v_1 t)\}, \ -\sin\{j(x-v_1 t)\}] \exp(\alpha j t), \tag{4.19}$$

where the vector $\boldsymbol{v}$ is defined by equation (4.6).

5. Explicit formulae for the tensor α_{rs}

It was asserted in § 3 that the equations (3.3), (3.4) and (3.6) have a unique solution $\boldsymbol{H}_0(\boldsymbol{x})$ for any given complex $\boldsymbol{H}_0^{\mathrm{A}}$, for all positive values of λ except possibly a bounded set of discrete values, possibly denumerable but with no non-zero point of accumulation. This is established below, and explicit expressions for both $\boldsymbol{H}_0(\boldsymbol{x})$ and the corresponding 3×3 tensor α_{rs} are obtained. The second is then used to establish a precise sense in which nearly all spatially periodic motions give dynamo action for nearly all resistivities λ.

Equation (3.3) with $p_0 = 0$, can be written as

$$\mathscr{T}\boldsymbol{H}_0' = -\nabla\times(\boldsymbol{u}\times\boldsymbol{H}_0^{\mathrm{A}}), \tag{5.1}$$

where $\mathscr{T}$ is the linear differential operator on the space V' of oscillatory solenoidal $\boldsymbol{u}$-periodic vector functions $\boldsymbol{H}'(\boldsymbol{x})$ defined by

$$\mathscr{T}\boldsymbol{H}' = \nabla\times(\boldsymbol{u}\times\boldsymbol{H}') + \lambda\nabla^2\boldsymbol{H}'. \tag{5.2}$$

The spectral properties of $\mathscr{T}$ are like those of $\mathscr{P}(\boldsymbol{j})$, and are described in appendix B. It is clear that equation (5.1) is uniquely soluble if $\mathscr{T}$ is invertible, or, equivalently since the eigenvalues of $\mathscr{T}$ have no point of accumulation, if zero is not an eigenvalue of $\mathscr{T}$.

Now $\mathscr{T}\boldsymbol{H}'$ can be written as

$$\mathscr{T}\boldsymbol{H}' = \nabla^2(\lambda\mathscr{I}-\mathscr{Q})\,\boldsymbol{H}', \tag{5.3}$$

where $\mathscr{I}$ is the identity operator and $\mathscr{Q}$ is another linear operator on the space V', defined by

$$\mathscr{Q}\boldsymbol{H}' = (\nabla^2)^{-1}\{-\nabla\times(\boldsymbol{u}\times\boldsymbol{H}')\}. \tag{5.4}$$

Thus $\mathscr{T}$ has the inverse $\mathscr{T}^{-1}$, given by

$$\mathscr{T}^{-1}\boldsymbol{H}' = (\lambda\mathscr{I}-\mathscr{Q})^{-1}(\nabla^2)^{-1}\boldsymbol{H}', \tag{5.5}$$

if and only if the operator $(\lambda\mathscr{I}-\mathscr{Q})$ is invertible.

It is shown in appendix B that the spectrum of $\mathscr{Q}$ is bounded and discrete, with no non-zero point of accumulation. Thus the resolvent $(\lambda\mathscr{I}-\mathscr{Q})^{-1}$ exists as a bounded operator for λ in the resolvent set, the complement of the spectrum, and is an analytic function of λ (see Dunford & Schwartz 1958, p. 566). This establishes the assertion of § 3.

From equations (5.1) and (5.5),

$$\boldsymbol{H}_0'(\boldsymbol{x}) = (\lambda\mathscr{I}-\mathscr{Q})^{-1}(\nabla^2)^{-1}\{-\nabla\times(\boldsymbol{u}\times\boldsymbol{H}_0^{\mathrm{A}})\}, \tag{5.6}$$

and the real tensor α_{qr} is now determined by equation (3.9):

$$\{(\boldsymbol{u}\times\boldsymbol{H}_0')^{\mathrm{A}}\}_q = \alpha_{qr}\{\boldsymbol{H}_0^{\mathrm{A}}\}_r. \tag{5.7}$$

It was shown in § 4 that a motion will give dynamo action for given positive resistivity λ if the real determinant

$$\alpha = |\alpha_{\overline{qr}}| \tag{5.8}$$

is non-zero. Now from equations (5.6) and (5.7), with the analyticity in λ of the resolvent

$(\lambda\mathscr{I}-\mathscr{Q})^{-1}$, α is an analytic function of λ, with poles at the discrete points of the spectrum of $\mathscr{Q}$. Thus if α has zeros on the real axis with a non-zero point of accumulation, it vanishes identically. So to prove dynamo action for a given motion and for nearly all resistivities it is sufficient to prove that α is non-zero for a single resistivity λ, or even that α is non-zero for sufficiently large λ.

Let q be the bound of the spectrum of $\mathscr{Q}$. Then for $\lambda > q$, the resolvent $(\lambda\mathscr{I}-\mathscr{Q})^{-1}$ can be written

$$(\lambda\mathscr{I}-\mathscr{Q})^{-1} = \sum_{n=0}^{\infty} \mathscr{Q}^n/\lambda^{n+1}, \tag{5.9}$$

where the series is convergent in the uniform operator topology (Dunford & Schwartz 1958, p. 567). Thus from equations (5.6) and (5.9),

$$\boldsymbol{H}_0'(\boldsymbol{x}) = \sum_{n=0}^{\infty} \mathscr{Q}^n(\nabla^2)^{-1}\{-\nabla\times(\boldsymbol{u}\times\boldsymbol{H}_0^{\mathrm{A}})\}/\lambda^{n+1}. \tag{5.10}$$

In order to apply this result, the Fourier series form (2.8) for $\boldsymbol{u}$ and $\boldsymbol{H}'$ must be used. From equation (5.4) the corresponding form for $\mathscr{Q}$ is given by

$$\{\mathscr{Q}\hat{\boldsymbol{H}}'\}(\boldsymbol{m}) = (\boldsymbol{m}^2)^{-1}\mathrm{i}\boldsymbol{m}\times\sum_{k\in K}\hat{\boldsymbol{u}}(\boldsymbol{k})\times\hat{\boldsymbol{H}}'(\boldsymbol{m}-\boldsymbol{k}), \tag{5.11}$$

for $\boldsymbol{m}$ in the set K defined by equation (2.9). Thus

$$(\boldsymbol{u}\times\boldsymbol{H}_0')^{\mathrm{A}} = \sum_{l=2}^{\infty}(\mathrm{i}/\lambda)^{l-1}\sum_{(l)}\boldsymbol{u}_l\times(\boldsymbol{s}_{l-1}\times(\boldsymbol{u}_{l-1}\times\ldots(\boldsymbol{s}_1\times(\boldsymbol{u}_1\times\boldsymbol{H}_0^{\mathrm{A}}))\ldots)), \tag{5.12}$$

where $\boldsymbol{u}_i = \hat{\boldsymbol{u}}(\boldsymbol{k}_i)$, $\sum_{(l)}$ denotes a sum over all ordered sets (chains) $\{\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_l\}$ of l non-zero vectors in K, with zero sum and with non-zero partial sums $\boldsymbol{m}_i = \boldsymbol{k}_1+\boldsymbol{k}_2+\ldots+\boldsymbol{k}_i$, and $\boldsymbol{s}_i = \boldsymbol{m}_i/(\boldsymbol{m}_i)^2$.

Equations (5.10) and (5.12) can incidentally be obtained in the following equivalent way. Equation (3.3) for $\boldsymbol{H}_0'(\boldsymbol{x})$ can be written

$$\lambda\nabla^2\boldsymbol{H}_0'(\boldsymbol{x}) = -\nabla\times(\boldsymbol{u}\times\boldsymbol{H}_0') - \nabla\times(\boldsymbol{u}\times\boldsymbol{H}_0^{\mathrm{A}}). \tag{5.13}$$

Writing

$$\boldsymbol{H}_0'(\boldsymbol{x}) = \sum_{1}^{\infty}\boldsymbol{h}^{(n)}/\lambda^n, \tag{5.14}$$

and substituting in equation (5.13) gives

$$\left.\begin{aligned}\boldsymbol{h}^{(1)} &= -(\nabla^2)^{-1}\nabla\times(\boldsymbol{u}\times\boldsymbol{H}_0^{\mathrm{A}}),\\ \boldsymbol{h}^{(n+1)} &= -(\nabla^2)^{-1}\nabla\times(\boldsymbol{u}\times\boldsymbol{h}^{(n)}) = \mathscr{Q}\boldsymbol{h}^{(n)}.\end{aligned}\right\} \tag{5.15}$$

Equations (5.14) and (5.15) are together equivalent to equation (5.10).

Equation (5.12) is clearly in the form (5.7), and the tensor α_{qr} has been found explicitly as a power series convergent for $\lambda > q$. As $\lambda\to\infty$,

$$\alpha_{qr} \sim \alpha_{qr}^{(2)}/\lambda, \tag{5.16}$$

$$\alpha \sim \alpha^{(2)}/\lambda^3 = |\alpha_{qr}^{(2)}|/\lambda^3, \tag{5.17}$$

where $\alpha_{qr}^{(2)}$ is determined by the $l = 2$ term in equation (5.12). For an incompressible fluid, with

$$\nabla\cdot\boldsymbol{u} = 0,$$

it is given by

$$\alpha_{qr}^{(2)} = -4\sum_{(K)}(\boldsymbol{v}\times\boldsymbol{w})\cdot\boldsymbol{k}(\boldsymbol{k}^2)^{-2}k_qk_r, \tag{5.18}$$

where $\hat{\boldsymbol{u}}(\boldsymbol{k}) = \boldsymbol{v}+\mathrm{i}\boldsymbol{w}$, and $\sum_{(K)}$ denotes a sum over non-zero vectors in K in which if $\boldsymbol{k}$ is included, $-\boldsymbol{k}$ is not.

The norm on the linear space of infinitely differentiable motions with fixed spatial periodicity, given by

$$\|\boldsymbol{u}\|^2 = \frac{1}{\tau}\int_\tau \boldsymbol{u}(\boldsymbol{x})^2\,\mathrm{d}V = \sum_K |\hat{\boldsymbol{u}}(\boldsymbol{k})|^2, \tag{5.19}$$

is now introduced. It is straightforward to show from equation (5.18), or, for compressible motions satisfying the equation of continuity, from the $l = 2$ term of equation (5.12), that the determinant $\alpha^{(2)}$ given by equation (5.17) is a continuous function of the motion with respect to this norm, and is non-zero on an open set with closure the whole space.

The main result can now be stated as follows. For any given spatial periodicity, there is a set of infinitely differentiable spatially periodic motions of an infinite homogeneous conducting fluid, each of which will give dynamo action for all finite resistivities but possibly a discrete set with no non-zero point of accumulation. This set of motions is open with respect to the norm (5.19) on the space of such motions; its closure is the whole space.

6. Related work on spatially periodic dynamos

Two independent papers on this subject were presented at the same conference in 1967. G. O. Roberts (1969) presented analytic and numerical results concerning the dynamo action of two particular spatially periodic motions, functions of only two Cartesian coordinates. These results are extended, a general theory for such dynamos is presented, and results for two additional particular motions are given, in a further paper (G. O. Roberts 1970*a*). The second paper, by Childress (1969), presented results given in detail in a report (Childress 1967*a*). These are now described in relation to, and in terms of, the present work.

Childress considers spatially periodic motions as described in § 2, with the minor restriction that the vectors $\boldsymbol{l}_1$, $\boldsymbol{l}_2$ and $\boldsymbol{l}_3$ are orthogonal. He requires that only a finite number of the Fourier series terms $\hat{\boldsymbol{u}}(\boldsymbol{k})$ are non-zero, cf. equation (2.8), and puts a further restriction on this set of $\boldsymbol{k}$ values, as mentioned below.

He confines attention to steady dynamos, with the added condition that the field tends to zero at infinity. It is possible to superpose field solutions of the form

$$\boldsymbol{B} = \boldsymbol{H}(\boldsymbol{x})\exp(\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}) \tag{6.1}$$

(where $\boldsymbol{H}(\boldsymbol{x})$ is $\boldsymbol{u}$-periodic (cf. equation (2.11)), to give a field vanishing at infinity, if and only if there are field solutions of the form (6.1) for $\boldsymbol{j}$ values on a curve with no subset of finite arc length lying identically in any plane, or on a surface with no subset of finite area lying identically in any plane (Lighthill 1960). Childress imposes the stronger condition that there be such solutions for all $\boldsymbol{j}$ in a closed convex analytic surface, in which case it is possible to superpose solutions so that $\boldsymbol{B} = O(r^{-1})$ at infinity. In any case, it is impossible to superpose solutions to obtain a steady field of finite total energy, as given by equations (1.4), (2.17) or (B 7).

In order to prove the existence of field solutions of the form (6.1), Childress considers a limit in which with fixed resistivity, the motion increases and its length scale decreases, while $\boldsymbol{j}$ in equation (6.1) above remains fixed in order of magnitude. With fixed motion as in this paper, his limit is equivalent to a simultaneous limit of large resistivity λ and small $\boldsymbol{j}$. This contrasts with the method in this paper of considering fixed resistivity of any magnitude, in a small $\boldsymbol{j}$ limit.

Childress obtains the same basic dynamo regeneration term of order $\boldsymbol{j}$ as that studied in this paper, and balances it with the natural decay rate $\lambda \boldsymbol{j}^2$ of a field of the form (6.1) to obtain a condition on $\boldsymbol{j}$. The exact equations are

$$\lambda \boldsymbol{j}^2 \boldsymbol{H}^{\mathrm{A}} = \mathrm{i}\boldsymbol{j} \times (\boldsymbol{u} \times \boldsymbol{H}')^{\mathrm{A}}, \tag{6.2}$$

$$-\lambda(\nabla + \mathrm{i}\boldsymbol{j})^2 \boldsymbol{H}' = (\nabla + \mathrm{i}\boldsymbol{j}) \times (\boldsymbol{u} \times \boldsymbol{H}^{\mathrm{A}}) + (\nabla + \mathrm{i}\boldsymbol{j}) \times (\boldsymbol{u} \times \boldsymbol{H}')', \tag{6.3}$$

cf. equations (2.12), (3.7) and (5.13). For sufficiently large λ, equation (6.3) can be solved just as equation (5.13) was, to obtain the power series (5.12) for $(\boldsymbol{u} \times \boldsymbol{H}')^{\mathrm{A}}$:

$$(\boldsymbol{u} \times \boldsymbol{H}')^{\mathrm{A}} = \sum_{l=2}^{\infty} \left(\frac{i}{\lambda}\right)^{l-1} \sum_{(l)} \boldsymbol{u}_l \times (\boldsymbol{s}_{l-1} \times (\boldsymbol{u}_{l-1} \times \ldots (\boldsymbol{s}_1 \times (\boldsymbol{u}_1 \times \boldsymbol{H}^{\mathrm{A}}))\ldots)), \tag{6.4}$$

where $\boldsymbol{s}_i$ is now $(\boldsymbol{m}_i + \boldsymbol{j})/(\boldsymbol{m}_i + \boldsymbol{j})^2$; the replacement of $\boldsymbol{m}_i$ in equation (5.12) by $\boldsymbol{m}_i + \boldsymbol{j}$ corresponds to the replacement of ∇ in equation (5.13) by $\nabla + \mathrm{i}\boldsymbol{j}$. Equation (6.4) defines the tensor α_{pq}, a complex function of λ and $\boldsymbol{j}$, and equation (6.2) can now be written as the eigenvalue problem

$$\lambda \boldsymbol{j}^2 b_q = \mathrm{i}\epsilon_{qrs} j_r \alpha_{sw} b_w,$$

cf. equation (3.11). The analysis of § 4 now applies without alteration, except that $\boldsymbol{v}$, α_{rs} and $\alpha^{\dagger}_{rs}$ are functions of $\boldsymbol{j}$ and are complex. So

$$\lambda \boldsymbol{j}^2 = \pm \sqrt{D} - \mathrm{i}\boldsymbol{j} \cdot \boldsymbol{v},$$

cf. equations (4.6), (4.9), (4.11) and (4.12), and

$$(\lambda \boldsymbol{j}^2 + \mathrm{i}\boldsymbol{j} \cdot \boldsymbol{v})^2 = D = \alpha^{\dagger}_{rs} j_r j_s. \tag{6.5}$$

This equation was obtained in an equivalent form by Childress. It is sufficient (though not necessary as he states) for the reality of equation (6.5) that $\boldsymbol{v}$ is imaginary and $\alpha_{\overline{rs}}$ real, and thus that α_{rs} is Hermitian. He proves that contributions to α_{rs} in equation (5.12) from even and odd values of l are respectively Hermitian and anti-Hermitian, and restricts the allowed non-zero Fourier components $\hat{\boldsymbol{u}}(\boldsymbol{k})$ so that there are no odd chains with the $\boldsymbol{u}_i$ all non-zero. He further shows from equation (6.4) that for $\boldsymbol{j} = 0$, α_{rs} is real (this fact is established in this paper basically from the reality of equation (5.13) which determines $\boldsymbol{H}'_0$ and α_{rs}). He confines attention (not explicitly) to motions for which this real tensor is non-zero, and asymptotically dominant for $\boldsymbol{j}$ non-zero.

Then in his limit
$$\alpha_{rs} \sim \alpha^{(0)}_{rs},$$

where $\alpha^{(0)}_{rs}$ is real and symmetric, with principal values α_1, α_2, α_3. So in the limit, equation (6.5) approximates to

$$(\lambda \boldsymbol{j}^2)^2 = j_1^2 \alpha_2 \alpha_3 + j_2^2 \alpha_3 \alpha_1 + j_3^2 \alpha_1 \alpha_2,$$

cf. equation (4.12). These equations define a bounded surface provided two of the α values have the same sign. The surface is convex everywhere provided that all the α values have the same sign and that the greatest is less than twice the least. Childress's result is only slightly different. He shows that in the limit the higher order terms in α_{rs} have small effect on the surface, so long as the reality of equation (6.5) is preserved.

This work is thus an advance from that of Childress in three main respects. First, a much wider class of motions is considered. The spatial periodicity is generalized, and motions periodic in space-time are included. There is no restriction on the number of Fourier components allowed in the series expansion of the motion. It is proved, effectively, that almost all motions give dynamo action. Secondly, there is no restriction on the resistivity; this is a major advance since low resistivity analysis and numerical work present particular difficulties. Thirdly, time-dependent

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dynamo action is included, and the asymptotic form of the growing field for a general initial field of finite energy is presented. All Childress's steady dynamos are unstable in the sense that his motions admit growing fields; furthermore, a steady-field solution cannot have finite energy.

Steenbeck, Krause & Rädler (1966) and Rädler (1969*a*) have studied the dynamo action of turbulence. They show that with the Lorentz force negligible, locally averaged quantities in a rotating turbulent fluid can satisfy the equation

$$\overline{\boldsymbol{u}'\times\boldsymbol{B}'} = \alpha\overline{\boldsymbol{B}}, \tag{6.6}$$

cf. equations (3.9) and (A 17). This result has been confirmed experimentally by Steenbeck, Kirko, Gailitis, Klawina, Krause, Lauminis & Lielausis (1967). In a limit where equation (6.6) is a satisfactory approximation, the equation for the locally averaged magetic field becomes

$$\partial\overline{\boldsymbol{B}}/\partial t = \nabla\times(\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}}+\alpha\overline{\boldsymbol{B}})+\lambda\nabla^2\overline{\boldsymbol{B}}, \tag{6.7}$$

and Steenbeck & Krause (1966, 1967, 1969) and Moffatt (1970) have studied this equation in a spherical geometry, looking for steady and oscillatory solutions, and applying the results to terrestrial, solar and stellar dynamos. The application of equations (6.6) and (6.7) is equivalent in a sense to a small $\boldsymbol{j}$ limit, with terms beyond the first order neglected. Rädler (1969*a*, *b*) has obtained, and studied the consequences of, alternatives to equation (6.6).

In a more rigorous application to finite regions, Childress (1967*b*) has proved that a spatially periodic motion, fitted into an insulated sphere by means of a cut-off function, can maintain a steady magnetic field. Essentially his method is to prove that the exact linear operator involved is very close to the simpler approximate linear operator derived using an equation corresponding to equation (6.7) above, in a low-conductivity limit where the size of the sphere becomes very large compared with the length scale of the spatial periodicity. It may be possible to generalize this finite region result to a wider class of motions and conductivities and to growing magnetic fields. A limit with the sphere large compared with the spatial periodicity length scale would again be required, so that the averaged equation (6.7) could be applied, with $\partial/\partial t$ replaced by a complex eigenvalue p and with the scalar α replaced by a tensor or a tensor function of position. It would only be necessary to prove that there was a solution of the exact equation close to a growing solution of the approximate equation.

In a future paper (G. O. Roberts 1970*b*), numerical results will be presented demonstrating the dynamo action of axisymmetric motions in a sphere. The motions $\boldsymbol{u}(r,\theta)$ in spherical polars are chosen to be closely analogous to a particular two-dimensional spatially periodic dynamo motion (G. O. Roberts 1970*a*), and growing field solutions of the form

$$\boldsymbol{B} = \boldsymbol{H}(r,\theta)\exp(pt+\mathrm{i}m\phi), \tag{6.8}$$

are found, cf. equation (2.11).

Appendices

A. *Motions periodic in space-time*

The analysis extending the results of §§ 2 to 5 to motions periodic in space-time is presented here, and the same result is obtained. If there is a fixed set of four independent space-time vectors $\{(\boldsymbol{l}_\theta, t_\theta): \theta = 1,2,3,4\}$ such that for all $\boldsymbol{x}$ and t, and for all θ values,

$$f(\boldsymbol{x}+\boldsymbol{l}_\theta, t+t_\theta) \equiv f(\boldsymbol{x},t), \tag{A 1}$$

then the function $f(\boldsymbol{x},t)$ is periodic in space-time. A function with the same periodicity as the motion $\boldsymbol{u}(\boldsymbol{x},t)$ will be called $\boldsymbol{u}$-periodic.

The average part f^A, and the oscillatory part $f'(\boldsymbol{x}, t)$ of such a function $f(\boldsymbol{x}, t)$ are defined in a way analogous to that for spatially periodic functions given in § 2:

$$f^A = \int_0^1 d\xi_1 \int_0^1 d\xi_2 \int_0^1 d\xi_3 \int_0^1 d\xi_4 f\left(\sum_1^4 \boldsymbol{l}_\theta \xi_\theta, \sum_1^4 t_\theta \xi_\theta\right)$$

$$= \frac{1}{\tau_4}\int_{\tau_4} f(\boldsymbol{x}, t)\, d\tau_4 \tag{A 2}$$

(where τ_4 is the space-time volume defined by the first integral), and

$$f'(\boldsymbol{x}, t) = f(\boldsymbol{x}, t) - f^A. \tag{A 3}$$

From these definitions, trivially,

$$(f')^A = 0, \tag{A 4}$$

$$(\partial f/\partial t)^A = 0, \tag{A 5}$$

$$(\nabla f)^A = 0. \tag{A 6}$$

Incidentally, motions which are periodic in time and space separately, for which a suitable choice of the four independent vectors has $t_1 = t_2 = t_3 = 0$ and $\boldsymbol{l}_4 = 0$, are a particular case of motions periodic in space-time. The analysis here applies to a much more general class of motions.

It is assumed below that $\boldsymbol{u}^A$ is zero. If this is not so in the original coordinate system, then it will be so with respect to moving axes, and the motion will remain periodic in space-time after such a Galilean transformation.

Functions periodic in space-time can be Fourier analysed in a way analogous to that given in § 2. The result is

$$f(\boldsymbol{x}, t) = \sum_{(\boldsymbol{k}, \omega)\in K_4} \hat{f}(\boldsymbol{k}, \omega) \exp\{i(\boldsymbol{k}\cdot\boldsymbol{x} + \omega t)\}, \tag{A 7}$$

where K_4 is the set of space-time vectors defined by

$$K_4 = \{(\boldsymbol{k}, \omega) : (\boldsymbol{k}, \omega) = 2\pi (L_4^T)^{-1}\boldsymbol{n};\ n_\theta \text{ integers}\}, \tag{A 8}$$

L_4^T being the transpose of the matrix L_4 with its columns the four independent space-time vectors.

Attention will be confined to magnetic fields of a form analogous to that given by equation (2.11):

$$\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{H}(\boldsymbol{x}, t) \exp(pt + i\boldsymbol{j}\cdot\boldsymbol{x}), \tag{A 9}$$

where $\boldsymbol{H}(\boldsymbol{x}, t)$ is $\boldsymbol{u}$-periodic, p and $\boldsymbol{H}$ are complex, and $\boldsymbol{j}$ is real. For spatially periodic motions, it was proved rigorously in § 2 that there is dynamo action in the sense that a finite initial magnetic energy can grow exponentially, if and only if the field equation (1.1) has a solution of the form (2.11) with the real part of p positive, for some $\boldsymbol{j}$. It will be assumed here that the same applies to motions periodic in space-time. Also, attention will be again confined to small $\boldsymbol{j}$.

Substitution of equation (A 9) into equations (1.1) and (1.2) gives

$$p\boldsymbol{H}(\boldsymbol{x}, t) = \mathscr{P}_4 \boldsymbol{H}(\boldsymbol{x}, t), \tag{A 10}$$

$$(\nabla + i\boldsymbol{j})\cdot\boldsymbol{H}(\boldsymbol{x}, t) = 0, \tag{A 11}$$

where $\mathscr{P}_4$ is the linear differential operator on the space $V_4(\boldsymbol{j})$ of $\boldsymbol{u}$-periodic vector functions satisfying equation (A 11), given by

$$\mathscr{P}_4 \boldsymbol{H}(\boldsymbol{x}, t) = -\dot{\boldsymbol{H}} + (\nabla + i\boldsymbol{j}) \times (\boldsymbol{u} \times \boldsymbol{H}) + \lambda(\nabla + i\boldsymbol{j})^2 \boldsymbol{H}. \tag{A 12}$$

It is assumed as in § 3 that for small $\boldsymbol{j}$,

$$\left.\begin{aligned} p &= \sum_0^\infty p_n, \\ \boldsymbol{H} &= \sum_0^\infty \boldsymbol{H}_n(\boldsymbol{x}, t), \end{aligned}\right\} \tag{A 13}$$

where $\boldsymbol{H}_n$ and p_n are explicitly of nth order in the components of $\boldsymbol{j}$. The justification of these series closely follows the degenerate perturbation theory described in appendix D. The analysis now follows that of § 3 closely. To zero order, equation (A 10) has the average

$$p_0 \boldsymbol{H}_0^{\mathrm{A}} = 0, \tag{A 14}$$

and attention is confined to solutions with $\boldsymbol{H}_0^{\mathrm{A}}$ non-zero. The oscillatory part of equation (A 10) is

$$\mathscr{T}_4 \boldsymbol{H}_0' = -\nabla \times (\boldsymbol{u} \times \boldsymbol{H}_0^{\mathrm{A}}), \tag{A 15}$$

where the operator $\mathscr{T}_4$ on the space V_4' of oscillatory $\boldsymbol{u}$-periodic vector functions $\boldsymbol{H}'(\boldsymbol{x}, t)$ is defined by the equation

$$\mathscr{T}_4 \boldsymbol{H}'(\boldsymbol{x}, t) = -\dot{\boldsymbol{H}}' + \nabla \times (\boldsymbol{u} \times \boldsymbol{H}') + \lambda \nabla^2 \boldsymbol{H}'. \tag{A 16}$$

Equation (A 15), with the assumption that the inverse $\mathscr{T}_4^{-1}$ exists as an analytic function of λ, determines $\boldsymbol{H}_0'$, and thus also the real tensor α_{qr}, again an analytic function of λ, such that

$$\{(\boldsymbol{u}' \times \boldsymbol{H}_0')^{\mathrm{A}}\}_q = \alpha_{qr} \{\boldsymbol{H}_0^{\mathrm{A}}\}_r. \tag{A 17}$$

To first order in the components of $\boldsymbol{j}$, equations (A 10) and (A 11) yield

$$p_1 \boldsymbol{H}_0^{\mathrm{A}} = \mathrm{i}\boldsymbol{j} \times (\boldsymbol{u} \times \boldsymbol{H}_0')^{\mathrm{A}}, \tag{A 18}$$

$$\boldsymbol{j} \cdot \boldsymbol{H}_0^{\mathrm{A}} = 0, \tag{A 19}$$

and so the analysis of § 4 applies exactly, showing that it is sufficient for dynamo action that the determinant

$$\alpha = |\alpha_{\overline{qr}}|, \tag{A 20}$$

an analytic function of λ, is non-zero.

It remains to analyse the operator $\mathscr{T}_4$. The operator $\mathscr{Q}_4$ on the space V_4' is defined by

$$\mathscr{Q}_4 \boldsymbol{H}'(\boldsymbol{x}, t) = \left(-\frac{\partial}{\partial t} + \lambda \nabla^2\right)^{-1} \{-\nabla \times (\boldsymbol{u} \times \boldsymbol{H}')\}, \tag{A 21}$$

so that

$$\mathscr{T}_4 \boldsymbol{H}'(\boldsymbol{x}, t) = \left(-\frac{\partial}{\partial t} + \lambda \nabla^2\right) (\mathscr{I} - \mathscr{Q}_4)\, \boldsymbol{H}', \tag{A 22}$$

and

$$\mathscr{T}_4^{-1} \boldsymbol{H}'(\boldsymbol{x}, t) = (\mathscr{I} - \mathscr{Q}_4)^{-1} \left(-\frac{\partial}{\partial t} + \lambda \nabla^2\right)^{-1} \boldsymbol{H}', \tag{A 23}$$

assuming that the operator $(\mathscr{I} - \mathscr{Q}_4)$ is invertible (cf. equations (5.3), (5.4) and (5.5)). Now the operator $\mathscr{Q}_4(\lambda)$ is compact, as shown in appendix B. So $(\mathscr{I} - \mathscr{Q}_4(\lambda))$ either has no inverse for any value of λ, or this inverse exists for all values of λ but possibly a discrete set with no non-zero point of accumulation (see Dunford & Schwartz (1958, p. 592)). The analysis below shows that $\mathscr{I} - \mathscr{Q}_4$ is invertible for sufficiently large λ, so the first alternative cannot apply. Further, the same lemma shows that if $\mathscr{T}_4(\lambda)^{-1}$ exists, it is analytic in λ.

Finally, the inverse $\mathscr{T}_4^{-1}$ can be obtained as an infinite series for large values of λ. In the Fourier series form

$$\{\hat{\mathscr{Q}}_4 \boldsymbol{H}'\}(\boldsymbol{m}, \mu) = (\mathrm{i}\mu + \lambda \boldsymbol{m}^2)^{-1} \mathrm{i}\boldsymbol{m} \times \sum_{(\boldsymbol{k}, \omega) \in K_4} \hat{\boldsymbol{u}}(\boldsymbol{k}, \omega) \times \hat{\boldsymbol{H}}(\boldsymbol{m} - \boldsymbol{k}, \mu - \omega), \tag{A 24}$$

for $(\boldsymbol{m}, \mu) \in K_4$. Hence, cf. equation (5.12),

$$(\boldsymbol{u} \times \boldsymbol{H}_0')^{\mathrm{A}} = \sum_{l=2}^{\infty} \sum_{(l)} \boldsymbol{u}_l \times (\boldsymbol{\theta}_{l-1} \times (\boldsymbol{u}_{l-1} \times \dots (\boldsymbol{\theta}_1 \times (\boldsymbol{u}_1 \times \boldsymbol{H}_0^{\mathrm{A}}))\dots)), \tag{A 25}$$

where $\boldsymbol{u}_i = \hat{\boldsymbol{u}}(\boldsymbol{k}_i, \omega_i)$, $\boldsymbol{\theta}_i = \mathrm{i}\boldsymbol{m}_i/(\mathrm{i}\mu + \lambda \boldsymbol{m}_i^2)$, and $\sum_{(l)}$ denotes a sum over all ordered sets

$$\{(\boldsymbol{k}_1, \omega_1), (\boldsymbol{k}_2, \omega_2), \dots, (\boldsymbol{k}_l, \omega_l)\}$$

of l non-zero vectors in the set K_4, with the sum $(\boldsymbol{m}_i, \mu_i) = (\boldsymbol{k}_1, \omega_1) + (\boldsymbol{k}_2, \omega_2) + \dots + (\boldsymbol{k}_i, \omega_i)$ non-zero for $i = 1, 2, \dots, l-1$, and zero for $i = l$. The series is convergent for sufficiently large λ, and equation (A 25) is in the form of equation (A 17). Further, as $\lambda \to \infty$,

$$\alpha_{qr} \sim \alpha_{qr}^{(2)}/\lambda, \tag{A 26}$$

$$\alpha \sim \alpha^{(2)}/\lambda^3 = |\alpha_{qr}^{(2)}|/\lambda^3, \tag{A 27}$$

where $\alpha_{qr}^{(2)}$ is determined by the $l = 2$ term in equation (A 25), and, for incompressible motions,

$$\alpha_{qr}^{(2)} = -4 \sum_{(K_4)} (\boldsymbol{v} \times \boldsymbol{w}) \,.\, \boldsymbol{k} (\boldsymbol{k}^2)^{-2} k_q k_r, \tag{A 28}$$

where $\hat{\boldsymbol{u}}(\boldsymbol{k}, \omega) = \boldsymbol{v} + \mathrm{i}\boldsymbol{w}$, and $\sum_{(K_4)}$ denotes a sum over non-zero vectors in K_4 in which if $(\boldsymbol{k}, \omega)$ is included, $(-\boldsymbol{k}, -\omega)$ is not.

The norm on the linear space of infinitely differentiable motions with fixed periodicity in space-time, given by

$$\|\boldsymbol{u}\|^2 = \frac{1}{\tau_4} \int_{\tau_4} \boldsymbol{u}(\boldsymbol{x}, t)^2 \, \mathrm{d}\tau_4$$
$$= \sum_{K_4} |\hat{\boldsymbol{u}}(\boldsymbol{k}, \omega)|^2, \tag{A 29}$$

is now introduced. The determinant $\alpha^{(2)}$, given by equations (A 27) and either (A 25) or (A 28), is then a continuous function of the motion, non-zero on an open set with closure the whole space. The result stated at the end of § 5, suitably reworded for motions periodic in space-time and for the norm (A 29), then follows.

B. *Spectral theory of the linear operators*

(*a*) *Results for* $\mathscr{P}(\boldsymbol{j})$ *and* $\mathscr{T}$

The linear differential operator $\mathscr{P}(\boldsymbol{j})$ on the space $V(\boldsymbol{j})$ of $\boldsymbol{u}$-periodic vector functions satisfying equation (2.13) is defined by equation (2.14). The linear differential operator $\mathscr{T}$ on the space V' of oscillatory solenoidal $\boldsymbol{u}$-periodic vector functions is defined by equation (5.2) and is closely related to $\mathscr{P}(\boldsymbol{0})$, as shown below. The results stated here for the operator $\mathscr{L}$ on the space V apply to both $\mathscr{P}(\boldsymbol{j})$ and $\mathscr{T}$, and also to the operator $\mathscr{P}_E(\boldsymbol{j})$ introduced below.

The eigenvalues l_n of $\mathscr{L}$ are complex in general, since $\mathscr{L}$ is not self-adjoint. Writing them as $\xi_n + \mathrm{i}\eta_n$, we have

$$\xi_n \to -\infty, \tag{B 1}$$

and there are constants ξ, c such that

$$|\eta_n| < c(\xi - \xi_n)^{\frac{1}{2}}. \tag{B 2}$$

Thus the eigenvalues are confined within a parabola with its axis coinciding with the negative real axis in the complex plane, and have no point of accumulation.

If all the eigenvalues are distinct, the corresponding eigenfunctions are complete in the space V. If there is degeneracy, this is not so in general, unless generalized eigenfunctions are included.

These are defined recursively as follows. $\boldsymbol{H}_{n,1}$ is a first-order eigenfunction of $\mathscr{L}$ with eigenvalue l_n if

$$(\mathscr{L}-l_n\mathscr{I})\,\boldsymbol{H}_{n,1}=0. \tag{B 3}$$

For $i>1$, $\boldsymbol{H}_{n,i}$ is an ith order eigenfunction of $\mathscr{L}$ with eigenvalue l_n if

$$(\mathscr{L}-l_n\mathscr{I})\,\boldsymbol{H}_{n,i}=\boldsymbol{H}_{n,i-1}, \tag{B 4}$$

where $\boldsymbol{H}_{n,i-1}$ is an $(i-1)$th order eigenfunction. Then there is an infinite set of independent generalized eigenvectors of $\mathscr{L}$ complete in V. They are all infinitely differentiable.

With respect to such a set as base, $\mathscr{L}$ has an infinite Jordan canonical matrix form. For

$$\mathscr{L}\left(\sum_{j=1}^{m_n}a_{n,j}\boldsymbol{H}_{n,j}\right)=\sum_{l=1}^{m_n}\left(\sum_{j=1}^{m_n}L_{ij}^{(n)}a_{n,j}\right)\boldsymbol{H}_{n,i}, \tag{B 5}$$

where $L^{(n)}$ is the $m_n\times m_n$ matrix with diagonal terms l_n and with the terms immediately above the diagonal unity, the other terms being all zeros. Thus this result is related to the spectral theory of non-Hermitian matrices of finite dimension.

Properly speaking, the result is that the set of generalized eigensolutions is complete in both of the Hilbert spaces V_0 and V_1, defined as the closures of the space of twice differentiable complex $\boldsymbol{u}$-periodic vector functions (with the relevant further restrictions for $V(\boldsymbol{j})$ or V'), with respect to the norms corresponding to the inner products

$$(\boldsymbol{f},\boldsymbol{g})_0=\frac{1}{\tau}\int_\tau \boldsymbol{f}\cdot\bar{\boldsymbol{g}}\,\mathrm{d}V, \tag{B 6}$$

$$(\boldsymbol{f},\boldsymbol{g})_1=\frac{1}{\tau}\int_\tau\left\{\boldsymbol{f}\cdot\bar{\boldsymbol{g}}+\frac{\partial f_i}{\partial x_j}\frac{\partial \bar{g}_i}{\partial x_j}\right\}\mathrm{d}V, \tag{B 7}$$

respectively, where the bar denotes the complex conjugate and the integration is over a single $\boldsymbol{u}$-periodicity cell (cf. equation (2.2)). Completeness means that finite linear combinations of the eigenfunctions are dense in the Hilbert space.

(*b*) *The operator* $\mathscr{P}_E(\boldsymbol{j})$

The operator $\mathscr{P}_E(\boldsymbol{j})$ on the extended space V_E of $\boldsymbol{u}$-periodic vector functions, without the restriction (2.13), is defined by the same equation as that defining $\mathscr{P}(\boldsymbol{j})$, equation (2.14).

The spectral theory of such elliptic linear differential operators was considered by Browder (1953). His analysis was for a single dependent variable vanishing on the boundary of a finite region. As he pointed out, this analysis can readily be extended to an operator such as $\mathscr{P}_E(\boldsymbol{j})$, with three dependent variables, the components of $\boldsymbol{H}$. The analysis is apparently slightly easier with the boundary condition of $\boldsymbol{u}$-periodicity which applies here; the difference is in the study of a Green function.

Browder's result for the operator $\mathscr{P}_E$ is precisely that stated above for $\mathscr{L}$. The result is given in Dunford & Schwartz (1963, p. 1746).

(*c*) *The operator* $\mathscr{P}(\boldsymbol{j})$

In order to establish the same result for the operator $\mathscr{P}(\boldsymbol{j})$, it must be shown that an independent subset of the generalized eigenvectors of $\mathscr{P}_E(\boldsymbol{j})$ are complete in $V(\boldsymbol{j})$, a closed subspace of V_E.

This depends on the following three lemmas, for which $\{p_n, \boldsymbol{H}_{n,1},\ldots,\boldsymbol{H}_{n,m_n}\}$ denotes the complete set of generalized eigensolutions of $\mathscr{P}_E(\boldsymbol{j})$. Note that the p_n are assumed distinct; the restatement and proof of the third lemma for the general case are tedious.

(i) Unless π_0 is in the known spectrum of the operator $\lambda(\nabla+\mathrm{i}\boldsymbol{j})^2$ on $\boldsymbol{u}$-periodic scalars, or the spectrum of $\mathscr{P}_E(\boldsymbol{j})$, the function

$$(\mathscr{P}_E(\boldsymbol{j})-\pi_0\mathscr{I})^{-1}\boldsymbol{H}$$

exists, and is in $V(\boldsymbol{j})$ for all $\boldsymbol{H}$ in $V(\boldsymbol{j})$.

(ii) For $i>1$,

$$(\mathscr{P}_E(\boldsymbol{j})-\pi_0\mathscr{I})^{-1}\boldsymbol{H}_{n,i}-(p_n-\pi_0)^{-1}\boldsymbol{H}_{n,i}=\boldsymbol{G}_{n,i-1}=-\sum_{j=2}^{m}(\pi_0-p_n)^{-j}\boldsymbol{H}_{n,i+1-j}.$$

(iii) If in the complete set of generalized eigenvectors of $\mathscr{P}_E(\boldsymbol{j})$, there are m_n associated independent generalized eigenvectors corresponding to the eigenvalue p_n, and μ_n of these are in $V(\boldsymbol{j})$, they are the first μ_n.

Suppose there is a vector function $\boldsymbol{H}_0$ in $V(\boldsymbol{j})$ with, in its expansion, non-vanishing coefficients of generalized eigenfunctions corresponding to eigenvalue p_0 which are not in $V(\boldsymbol{j})$. Choose a π_0 satisfying the above conditions and nearer to p_0 than to any other of the discrete eigenvalues of $\mathscr{P}_E(\boldsymbol{j})$. Operate successively according to (ii) above to eliminate any eigensolutions $\boldsymbol{H}_{0,i}$ of higher order than μ_0+1. Subtract out the first μ_0 eigensolutions, to obtain the function $\boldsymbol{H}_1$, still in $V(\boldsymbol{j})$. Now subtract out the component of $\boldsymbol{G}_{0,\mu_0}$ from

$$(p_0-\pi_0)(\mathscr{P}_E(\boldsymbol{j})-\pi_0\mathscr{I})^{-1}\boldsymbol{H}_1,$$

to obtain the function $\boldsymbol{H}_2$, still in $V(\boldsymbol{j})$. Repeat to obtain a sequence of functions $\boldsymbol{H}_n$ in $V(\boldsymbol{j})$ and converging to a non-zero multiple of $\boldsymbol{H}_{0,\mu_0+1}$, which is not in $V(\boldsymbol{j})$. But $V(\boldsymbol{j})$ is closed, hence there is a contradiction.

(d) *The operator* $\mathscr{T}$

The space $V(\boldsymbol{0})$ is the Cartesian product of the space V' of functions $\boldsymbol{H}'$ (on which $\mathscr{T}$ operates) and the space of complex vectors $\boldsymbol{H}^{\Delta}$. The operator $\mathscr{P}(\boldsymbol{0})$ on $V(\boldsymbol{0})$ is given by

$$\mathscr{P}(\boldsymbol{0})\,\boldsymbol{H}=\mathscr{P}(\boldsymbol{0})\,(\boldsymbol{H}^{\Delta}+\boldsymbol{H}')$$
$$=\nabla\times(\boldsymbol{u}\times\boldsymbol{H}^{\Delta})+\mathscr{T}\boldsymbol{H}'.$$

The proof that the operator $\mathscr{T}$ on V' has the same spectral properties as those described for $\mathscr{L}$ above depends on the following lemmas.

(i) Any generalized eigenfunction $\boldsymbol{H}_{n,i}$ of $\mathscr{P}(\boldsymbol{0})$ with non-zero average corresponds to an eigenvalue $p_n=0$, and is the highest order generalized eigenfunction of its set, i.e. $i=m_n$.

(ii) If necessary, the set of independent generalized eigenfunctions of $\mathscr{P}(\boldsymbol{0})$ complete in $V(\boldsymbol{0})$ can be re-chosen so that there are only three with non-vanishing average, these averages being independent.

The remaining eigenfunctions are then also generalized eigenfunctions of the operator $\mathscr{T}$ on V', and are complete in V'.

(e) *Continuity of the spectrum of* $\mathscr{P}(\boldsymbol{j})$

The spectrum of $\mathscr{P}_E(\boldsymbol{j})$ is a continuous function of λ, the Fourier components $\hat{\boldsymbol{u}}(\boldsymbol{k})$ of the motion, and of $\boldsymbol{j}$. The same applies to the spectrum of $\mathscr{P}(\boldsymbol{j})$, except that there is a discontinuity at $\boldsymbol{j}=0$, one further eigenvalue zero is introduced because the condition (2.13) has its average an identity only when $\boldsymbol{j}$ is precisely zero.

Further, at isolated eigenvalues (where there is only a first-order eigenfunction, and the eigenvalue is not repeated) the eigenvalue is an infinitely differentiable function of these quantities, and the corresponding eigenvector can have its arbitrary complex multiple chosen so that it too is an infinitely differentiable function. These results are established by standard perturbation theory, see for example, Dunford & Schwartz (1958, p. 587), Kato (1966).

(f) *The operators $\mathscr{Q}$ and $\mathscr{Q}_4$*

The linear operators $\mathscr{Q}$ and $\mathscr{Q}_4$ are defined by equations (5.4) and (A 21), with the corresponding Fourier series forms (5.11) and (A 24) respectively. All that is needed in this paper is a knowledge of the spectrum of each.

The operator $\mathscr{Q}$ is the limit in the uniform operator topology of the operator $\mathscr{Q}(n)$ defined by

$$\{\widehat{\mathscr{Q}(n)}\,\boldsymbol{H}'\}(\boldsymbol{m}) = (\boldsymbol{m}^2)^{-1}\mathrm{i}\boldsymbol{m}\times\sum_{\boldsymbol{k}\in K}\hat{\boldsymbol{u}}(\boldsymbol{k})\times\hat{\boldsymbol{H}}(\boldsymbol{m}-\boldsymbol{k}) \quad \text{for} \quad |\boldsymbol{m}|\leqslant n,$$

and

$$\{\widehat{\mathscr{Q}(n)}\,\boldsymbol{H}'\}(\boldsymbol{m}) = 0 \quad \text{for} \quad |\boldsymbol{m}| > n.$$

Now $\mathscr{Q}(n)$ has finite dimensional range, and thus is compact. So $\mathscr{Q}$ is also compact (see Dunford & Schwartz 1958, p. 486). The operator $\mathscr{Q}_4$ can be shown to be compact in the same way, defining $\mathscr{Q}_4(n)$ to have the range $\boldsymbol{k}^2+\omega^2\leqslant n^2$.

Compact operators have a discrete bounded spectrum with no non-zero point of accumulation (see Dunford & Schwartz 1958, p. 579); on the complementary resolvent set the resolvent $(\lambda\mathscr{I}-\mathscr{Q})^{-1}$ is an analytic function of λ.

C. *The form of the growing magnetic field*

This appendix completes the analysis of § 2 in three points. First, it was assumed in obtaining equation (2.24) that the linear differential operator $\mathscr{P}$ has a complete set of eigenvectors for each value of $\boldsymbol{j}$. But as stated in appendix B, this is only so in general if generalized eigenvectors are included.

The field solution $\boldsymbol{B}(\boldsymbol{x},t)$ corresponding to a given initial $\boldsymbol{B}(\boldsymbol{x})$ can be Fourier analysed to a form corresponding to equation (2.21):

$$\boldsymbol{B}(\boldsymbol{x},t) = \int_J \boldsymbol{H}(\boldsymbol{x},\boldsymbol{j},t)\exp(\mathrm{i}\boldsymbol{j}\,.\,\boldsymbol{x})\,\mathrm{d}^3\boldsymbol{j}, \tag{C 1}$$

and the function $\boldsymbol{H}(\boldsymbol{x},\boldsymbol{j},t)$ is in $V(\boldsymbol{j})$ and can be expressed in terms of the generalized eigenfunctions of $\mathscr{P}$ in the form

$$\boldsymbol{H}(\boldsymbol{x},\boldsymbol{j},t) = \sum_{n=1}^{\infty}\sum_{i=1}^{m_n} a_{n,i}(\boldsymbol{j},t)\,\boldsymbol{H}_{n,i}(\boldsymbol{x},\boldsymbol{j}), \tag{C 2}$$

where m_n generalized eigenfunctions $\boldsymbol{H}_{n,i}(\boldsymbol{x},\boldsymbol{j})$ correspond to the eigenvalue $p_n(\boldsymbol{j})$, cf. equations (B 3), (B 4).

The m components $a_i(t)$ corresponding to a particular generalized eigenfunction set are determined by the equations

$$\left.\begin{aligned} \dot{a}_i &= pa_i + a_{i+1} \quad (i = 1, 2, \ldots, m-1),\\ \dot{a}_m &= pa_m, \end{aligned}\right\} \tag{C 3}$$

(cf. equation (B 5)), with the solution

$$\left.\begin{aligned} a_m\ \ \mathrm{e}^{-pt} &= b_m,\\ a_{m-1}\,\mathrm{e}^{-pt} &= b_m t + b_{m-1},\\ &\vdots\\ a_1\ \ \mathrm{e}^{-pt} &= \frac{b_m t^{m-1}}{(m-1)!} + \frac{b_{m-1}t^{m-2}}{(m-2)!} + \ldots + b_2 t + b_1, \end{aligned}\right\} \tag{C 4}$$

$b_1, b_2, \ldots, b_m$ being the initial values $a_1(0), a_2(0), \ldots, a_m(0)$. So the generalization of equation (2.24) is, from equations (C 1), (C 2) and (C 4),

$$\boldsymbol{B}(\boldsymbol{x}, t) = \int_J \mathrm{d}^3\boldsymbol{j} \sum_{n=1}^{\infty} \boldsymbol{H}_n(\boldsymbol{x}, \boldsymbol{j}, t) \exp(p_n t + \mathrm{i}\boldsymbol{j}\,.\,\boldsymbol{x}), \tag{C 5}$$

where

$$\boldsymbol{H}_n(\boldsymbol{x}, \boldsymbol{j}, t) = \sum_{i=1}^{m_n} a_{n,i}(\boldsymbol{j}, 0) \sum_{k=1}^{i} \frac{t^{i-k}}{(i-k)!} \boldsymbol{H}_{n,k}(\boldsymbol{x}, \boldsymbol{j}). \tag{C 6}$$

Secondly, in § 2 it was stated that a motion will give dynamo action, in the sense that a finite field energy can grow exponentially, if and only if there is for some $\boldsymbol{j}$ an eigenvalue of $\mathscr{P}$ with positive real part. Equation (2.24) or the more general result (C 5), do not establish this rigorously, and the energy equation (2.17) must be appealed to. This implies

$$E(t) = \pi^2 \int_J \mathrm{d}^3\boldsymbol{j} \sum_{k\in K} |\hat{\boldsymbol{B}}(\boldsymbol{j}+\boldsymbol{k})|^2$$
$$= \frac{\pi^2}{\tau} \int_J \mathrm{d}^3\boldsymbol{j} \int_\tau |\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{j}, t)|^2 \,\mathrm{d}\tau, \tag{C 7}$$

where τ is the volume defined by equation (2.2), and $\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{j}, t)$ is defined by equation (C 1), and expanded in generalized eigensolutions of $\mathscr{P}$ in equation (C 2). Using equation (C 2),

$$\frac{\pi^2}{\tau} \int_\tau |\boldsymbol{H}(\boldsymbol{x}, \boldsymbol{j}, t)|^2 \,\mathrm{d}\tau = \Sigma \overline{a_{n_1, i_1}} M_{n_1, i_1; n_2, i_2} a_{n_2, i_2}$$
$$= \boldsymbol{a}^* M \boldsymbol{a}, \tag{C 8}$$

where M is a constant Hermitian positive definite infinite matrix, the bar denotes complex conjugate, and $\boldsymbol{a}^*$ is the transposed conjugate of the column vector $\boldsymbol{a}(t)$ of complex amplitudes $a_{n,i}(t)$ given by equation (C 4). Thus

$$E(t) = \int_J \mathrm{d}^3\boldsymbol{j}\{\boldsymbol{a}^* M \boldsymbol{a}\}(\boldsymbol{j}). \tag{C 9}$$

Further, from the continuity property of the spectrum, stated in appendix B, $\boldsymbol{a}^* M \boldsymbol{a}$ is a continuous function of $\boldsymbol{j}$. Thus $E(t)$ can grow exponentially, indefinitely, if and only if one at least of the amplitudes $a_{n,i}(\boldsymbol{j}, t)$ can grow exponentially, that is, if and only if there is an eigenvalue p_n with positive real part.

It remains, thirdly, to prove the result (2.25) for the asymptotic form of the growing magnetic field for large time. Assume that the eigenvalue $p_1(\boldsymbol{j}_0)$ with the largest real part is non-degenerate and, further, that $\mathrm{Re}\{p_1(\boldsymbol{j})\}$ has a simple maximum at $\boldsymbol{j}_0$. Then $\boldsymbol{f}$, defined by equation (2.27), is real, and the real part of the matrix P defined by equation (2.28) is positive definite. Further, $a_1(\boldsymbol{j})\,\boldsymbol{H}_1(\boldsymbol{x}, \boldsymbol{j})$ is differentiable at $\boldsymbol{j}_0$. In the equation (2.24) or (C 5) for the growing magnetic field, asymptotically significant contributions will all come from values of $\boldsymbol{j}$ very near to $\pm \boldsymbol{j}_0$. The value of p_1 near $\boldsymbol{j}_0$ can be written

$$p_1(\boldsymbol{j}) = p_1(\boldsymbol{j}_0) - \mathrm{i}\boldsymbol{f}\,.\,\boldsymbol{l} - \boldsymbol{l}' P \boldsymbol{l} + O(l^3), \tag{C 10}$$

where $\boldsymbol{l}$ is $\boldsymbol{j} - \boldsymbol{j}_0$. Further, the contributions to $\boldsymbol{B}(\boldsymbol{x}, t)$ from $\pm \boldsymbol{j}_0$ are conjugate, so that $\boldsymbol{B}(\boldsymbol{x}, t)$ is real. So, using equation (2.24) or equation (C 5), equation (2.25) is obtained in the limit $t \to \infty$, with

$$I(\boldsymbol{x}, t) = \int_l \mathrm{d}^3\boldsymbol{l} \exp\{\mathrm{i}\boldsymbol{l}\,.\,(\boldsymbol{x} - \boldsymbol{f}t) - \boldsymbol{l}' P \boldsymbol{l} t\}, \tag{C 11}$$

where the integral can be taken as over all space; only $\boldsymbol{l}$ values very close to zero make a significant asymptotic contribution. It should be noted that the contribution from the derivative of $a_1(\boldsymbol{j})\,\boldsymbol{H}_1(\boldsymbol{x}, \boldsymbol{j})$ with respect to $\boldsymbol{j}$ at $\boldsymbol{j}_0$ is asymptotically negligible provided $a_1(\boldsymbol{j}_0)$ is non-zero.

The evaluation of the integral (C 11) in the form of equation (2.26), and the result (2.29), are difficult because P is complex. They are consequences of the following results.

(i) The inverse of a symmetric matrix with positive definite real part has positive definite real part. For let the matrix be $P_r + \mathrm{i}P_i$. Then its inverse is

$$(P_r + P_i P_r^{-1} P_i)^{-1} - \mathrm{i} P_r^{-1} P_i (P_r + P_i P_r^{-1} P_i)^{-1}.$$

(ii) For any non-singular $n \times n$ matrix, an r-rowed minor of the adjoint is the product of two terms, the cofactor of the corresponding r-rowed minor of the transpose, and the $(r-1)$th power of the determinant.

The rth compound matrix $A^{(r)}$ of a $k \times n$ matrix is the ${}_kC_r \times {}_nC_r$ matrix of r-rowed minors in dictionary order. By the Binet–Cauchy theorem, if $C = AB$ then $C^{(r)} = A^{(r)}B^{(r)}$. Denoting the matrix, adjoint, and determinant by P, P^{A}, p, we have

$$PP^{\mathrm{A}} = pI$$

whence
$$P^{(r)}(P^{\mathrm{A}})^{(r)} = (pI)^{(r)};$$

therefore
$$(P^{\mathrm{A}})^{(r)} = p^r (P^{(r)})^{-1}. \tag{C 12}$$

Now with K, L, M denoting ordered sets of r integers from 1 to n, and with K', L', M' denoting the corresponding ordered complementary sets of $n-r$ integers, by the Laplace expansion of a determinant,

$$\sum_L p_{KL} p_{M'L'} \eta_M \eta_L = p\,\delta_{KM},$$

where p_{KL} is the r-rowed minor, δ_{KM} has the obvious Kronecker meaning, and η_K is the sign of the permutation $\begin{Bmatrix} 1, \ldots, n \\ K, \quad K' \end{Bmatrix}$. But

$$\{P^{(r)}\}_{KL} = p_{KL}.$$

So
$$\begin{aligned} \{(P^{(r)})^{-1}\}_{LM} &= p^{-1} p_{M'L'} \eta_M \eta_L \\ &= p^{-1} \times \text{cofactor of } p_{ML} \\ &= p^{-1} \times \text{cofactor of } \{P^{\mathrm{T}}\}_{LM}, \end{aligned} \tag{C 13}$$

where the T denotes the transpose. Equations (C 12) and (C 13) together give the required result.

(iii)
$$\begin{aligned} I &= \int_{-\infty}^{\infty} \mathrm{d}x_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}x_n \exp\left(-P_{kl} x_k x_l + q_k x_k\right) \\ &= \pi^{\frac{1}{2}n} p^{-\frac{1}{2}} \exp\{\tfrac{1}{4}(P^{-1})_{kl} q_k q_l\}, \end{aligned} \tag{C 14}$$

where P is complex symmetric with inverse P^{-1} and determinant p, q_k is complex, and Re (P) is positive definite.

For real P this result is straightforward, since P is diagonal with respect to a new orthogonal coordinate system. For complex P the argument is more difficult, and proceeds by induction.

For $n = 1$,
$$\begin{aligned} &\int_{-\infty}^{\infty} \exp\left(-px^2 + qx\right) \mathrm{d}x \\ &\quad = \int_{-\infty}^{\infty} \exp\left\{-p\left(x - \frac{q}{2p}\right)^2\right\} \mathrm{d}x \exp\left(q^2/4p\right) \\ &\quad = \pi^{\frac{1}{2}} p^{-\frac{1}{2}} \exp\left(q^2/4p\right), \end{aligned}$$

by contour integration.

Now assume the theorem is true for dimension up to $n-1$. Use Latin suffixes for the numbers

1 to n, and Greek for 2 to n, with the summation convention for each, and denote the cofactors of the $(n-1)\times(n-1)$ matrix by $\pi^{A}_{\kappa\lambda}$. Then the integral (C 14) is

$$I = \int_{-\infty}^{\infty} \mathrm{d}x_1 \exp\,(p_{1,1}x_1^2+q_1x_1)\,\pi^{\frac{1}{2}(n-1)}\,(p^{A}_{1,1})^{-\frac{1}{2}}\exp\,\{\pi^{A}_{\kappa\lambda}(q_\kappa-2p_{\kappa 1}x_1)\,(q_\lambda-2p_{\lambda 1}x_1)/4p^{A}_{1,1}\},$$

using the theorem in $(n-1)$ dimensions, with x_1 regarded as a constant, the determinant being $p^{A}_{1,1}$ and q_k being replaced by $q_\kappa-2p_{\kappa 1}x_1$. Now use the theorem in one dimension to obtain

$$I = \pi^{\frac{1}{2}n}(p^{A}_{1,1})^{-\frac{1}{2}}\,(p_{1,1}-\pi^{A}_{\kappa\lambda}p_{\lambda 1}p_{\kappa 1}/p^{A}_{1,1})^{-\frac{1}{2}}\exp\left\{\frac{\pi^{A}_{\kappa\lambda}q_\kappa q_\lambda}{4p^{A}_{1,1}}+\frac{(g_1-p_{\kappa 1}q_\lambda\pi^{A}_{\kappa\lambda}/p^{A}_{1,1})^2}{4(p_{1,1}-p_{\kappa 1}p_{\lambda 1}\pi^{A}_{\kappa\lambda}/p^{A}_{1,1})}\right\},$$

which reduces to equation (C 14) after some effort, using (ii) above in the form

$$p^{A}_{1,1}p^{A}_{\kappa\lambda}-p^{A}_{1\lambda}p^{A}_{\kappa 1} = p\pi^{A}_{\kappa\lambda}.$$

Finally, the result (2.30) for the asymptotic total energy $E(t)$ follows from equation (C 7) and the assumptions in § 2. For, doubling the contribution from values of $\boldsymbol{j}$ near to $+\boldsymbol{j}_0$,

$$E(t) \sim \frac{2\pi^2}{\tau}\int_\tau \mathrm{d}\tau\int_l \mathrm{d}^3\boldsymbol{l}\,|a_1(\boldsymbol{j}_0)\,\boldsymbol{H}_1(\boldsymbol{x},\boldsymbol{j}_0)\exp\{p_1(\boldsymbol{j}_0+\boldsymbol{l})\,t\}|^2,$$

and the $\boldsymbol{l}$ integral can be asymptotically evaluated using equation (C 10).

D. *The perturbation method for small* $\boldsymbol{j}$

In § 3 the eigenvalue problem defined by equations (2.12) to (2.14) was studied by a perturbation method for small $\boldsymbol{j}$, and attention was confined to the first term p_1 of an expansion in ascending powers of the components of $\boldsymbol{j}$ for the eigenvalues p vanishing to zero order. This expansion is now justified for the main case considered, when the resistivity λ is such that the operator $\mathscr{T}$ defined by equation (5.2) is invertible (cf. § 5), and when the corresponding tensor α_{qr} and the direction $\hat{\boldsymbol{j}}$ of $\boldsymbol{j}$ are such that the quantity D defined by equation (4.12) is positive. It is also shown below how further terms in the expansion can be obtained.

Equation (2.12) implies that

$$\{p-\lambda(\nabla+\mathrm{i}\boldsymbol{j})^2\}\{(\nabla+\mathrm{i}\boldsymbol{j})\,.\,\boldsymbol{H}\} = 0. \tag{D 1}$$

Thus the restriction (2.13) is redundant provided that

$$p \neq -\lambda(\boldsymbol{j}+\boldsymbol{k})^2 \tag{D 2}$$

for any vector $\boldsymbol{k}$ in K. The average and oscillatory parts of equation (2.12) can be written in the following forms:

$$p^*\boldsymbol{H}^{A} = \mathrm{i}\boldsymbol{j}\times(\boldsymbol{u}\times\boldsymbol{H}')^{A} = \mathscr{R}_1\boldsymbol{H}', \tag{D 3}$$

$$\begin{aligned} p^*\,\boldsymbol{H}' &= (\nabla+\mathrm{i}\boldsymbol{j})\times(\boldsymbol{u}\times\boldsymbol{H}^{A})+(\nabla+\mathrm{i}\boldsymbol{j})\times(\boldsymbol{u}\times\boldsymbol{H}')'+\lambda\nabla^2\boldsymbol{H}'+2\mathrm{i}\lambda(\boldsymbol{j}\,.\nabla)\,\boldsymbol{H}' \\ &= (\mathscr{S}+\mathscr{S}_1)\,\boldsymbol{H}^{A}+(\mathscr{T}+\mathscr{T}_1)\,\boldsymbol{H}', \end{aligned} \tag{D 4}$$

where $p^* = p+\lambda\boldsymbol{j}^2$, the operators $\mathscr{S}$ and $\mathscr{T}$ are of zero order in the components of $\boldsymbol{j}$, the operators $\mathscr{R}_1$, $\mathscr{S}_1$, and $\mathscr{T}_1$ are of first order, and $\mathscr{T}$ is invertible by assumption.

Equations (D 3) and (D 4) define the operator $\mathscr{P}_E(j\hat{\boldsymbol{j}})$ on $\boldsymbol{u}$-periodic vector functions without restriction, cf. appendix B, § (*b*). For fixed $\hat{\boldsymbol{j}}$, $\mathscr{P}_E$ is an analytic operator-valued function of j. Further, for $j = 0$, zero is an eigenvalue, with multiplicity 3. Thus (see Dunford & Schwartz 1958, p. 587) for sufficiently small $|j|$, $\mathscr{P}_E(j\hat{\boldsymbol{j}})$ has the three eigenvalues $\lambda_1(j)$, $\lambda_2(j)$, $\lambda_3(j)$, analytic functions of the principal values of $j^{1/m}$, where $m = 1$, 2, or 3. The result required is that $m = 1$, so that the expansion (5) applies.

It is now assumed as in § 3 that an eigensolution can be expanded for sufficiently small real $\boldsymbol{j}$ as

$$p = \sum_0^\infty p_n, \quad p^* = \sum_0^\infty p_n^*, \quad \boldsymbol{H} = \sum_0^\infty \boldsymbol{H}_n, \tag{D 5}$$

where p_n, p_n^* and $\boldsymbol{H}_n$ are explicitly of nth order in the components of $\boldsymbol{j}$, and $p_n = p_n^*$ for $n \neq 2$. To zero order, equation (D 3) gives

$$p_0^* \boldsymbol{H}_0^{\mathrm{A}} = 0, \tag{D 6}$$

and attention is confined to eigensolutions with $\boldsymbol{H}_0^{\mathrm{A}}$ non-zero.

To zero and first order respectively, equations (D 4) and (D 3) give

$$0 = \mathscr{S}\boldsymbol{H}_0^{\mathrm{A}} + \mathscr{T}\boldsymbol{H}_1', \tag{D 7}$$

$$\begin{aligned} p_1^* \boldsymbol{H}_0^{\mathrm{A}} &= \mathscr{R}_1 \boldsymbol{H}_1' \\ &= -\mathscr{R}_1 \mathscr{T}^{-1} \mathscr{S} \boldsymbol{H}_0^{\mathrm{A}} \end{aligned} \tag{D 8}$$

and this is the eigenvalue problem studied in § 4; in fact

$$\{-\mathscr{R}_1 \mathscr{T}^{-1} \mathscr{S}\}_{ps} = \beta_{ps} \tag{D 9}$$

cf. equation (4.1). There are thus three distinct eigenvalues p_1^*, given by

$$\beta_1 = \sqrt{D - \mathrm{i}\boldsymbol{j}\cdot\boldsymbol{\nu}}, \quad \beta_2 = -\sqrt{D - \mathrm{i}\boldsymbol{j}\cdot\boldsymbol{\nu}}, \quad \beta_3 = 0, \tag{D 10}$$

with corresponding left and right eigenvectors $\boldsymbol{l}_i$, $\boldsymbol{r}_i$, functions of $\hat{\boldsymbol{j}}$, defined so that

$$|\boldsymbol{l}_i|^2 = 1, \quad \boldsymbol{l}_i \cdot \boldsymbol{r}_i = 1, \tag{D 11}$$

and therefore satisfying

$$\boldsymbol{l}_i \cdot \boldsymbol{r}_j = \delta_{ij}. \tag{D 12}$$

It should be noted that $\boldsymbol{l}_3$ is a multiple of $\hat{\boldsymbol{j}}$, as mentioned in § 4, and that

$$\boldsymbol{l}_3 \cdot \mathscr{R}_1 = 0. \tag{D 13}$$

For definiteness, take the growing solution

$$p_1^* = \beta_1, \quad \boldsymbol{H}_0^{\mathrm{A}} = \boldsymbol{r}_1. \tag{D 14}$$

To first and second order respectively, equations (D 4) and (D 3) give

$$\begin{aligned} \mathscr{T}\boldsymbol{H}_1' &= -\mathscr{S}\boldsymbol{H}_1^{\mathrm{A}} + p_1^* \boldsymbol{H}_0' - \mathscr{S}_1 \boldsymbol{H}_0^{\mathrm{A}} - \mathscr{T}_1 \boldsymbol{H}_0' \\ &= -\mathscr{S}\boldsymbol{H}_1^{\mathrm{A}} + \boldsymbol{G}_1', \end{aligned} \tag{D 15}$$

$$p_2^* \boldsymbol{H}_0^{\mathrm{A}} + p_1^* \boldsymbol{H}_1^{\mathrm{A}} = \mathscr{R}_1 \boldsymbol{H}_1', \tag{D 16}$$

where $\boldsymbol{G}_1'$ is now known, from equations (D 7) and (D 13). Thus

$$p_2^* \boldsymbol{H}_0^{\mathrm{A}} + (p_1^* + \mathscr{R}_1 \mathscr{T}^{-1} \mathscr{S})\, \boldsymbol{H}_1^{\mathrm{A}} = \mathscr{R}_1 \mathscr{T}^{-1} \boldsymbol{G}_1'. \tag{D 17}$$

There is an arbitrariness about $\boldsymbol{H}_1^{\mathrm{A}}$, since clearly if equation (D 17) is satisfied, it will remain so with the addition to $\boldsymbol{H}_1^{\mathrm{A}}$ of any multiple of $\boldsymbol{H}_0^{\mathrm{A}} = \boldsymbol{r}_1$. The same arbitrariness occurs at all orders, as usual in perturbation theory. So we impose

$$\boldsymbol{H}_r^{\mathrm{A}} = c_r \boldsymbol{r}_2 + d_n \boldsymbol{r}_3 \tag{D 18}$$

for $n \geqslant 1$, where c_n, d_n are explicitly of order n in the components of $\boldsymbol{j}$. Then multiplying equation (D 17) on the left by $\boldsymbol{l}_1$, $\boldsymbol{l}_2$, $\boldsymbol{l}_3$ and using equations (D 10), (D 12), (D 13), (D 14) and (D 18) gives

$$p_2^* = \boldsymbol{l}_1 \cdot \mathscr{R}_1 \mathscr{T}^{-1} \boldsymbol{G}_1', \tag{D 19}$$

$$(\beta_1 - \beta_2)\, c_1 = \boldsymbol{l}_2 \cdot \mathscr{R}_1 \mathscr{T}^{-1} \boldsymbol{G}_1', \tag{D 20}$$

$$(\beta_1 - \beta_3)\, d_1 = 0. \tag{D 21}$$

Thus $\boldsymbol{H}_1^{\mathrm{A}}$ is known from equation (D 18), and $\boldsymbol{H}_1'$ from equation (D 15).

The work of the preceding paragraph can be repeated at successively higher orders, with successive known $\boldsymbol{G}_n'$ in equations like equation (D 15), and with p_{n+1}^*, c_n, and $d_n = 0$ successively determined from equations like equation (D 17).

Finally, then, since p_1^* is non-zero, equation (D 2) is satisfied for sufficiently small $\boldsymbol{j}$, justifying the neglect of condition (2.13). The form of the result shows that $m = 1$ in the analysis of $\mathscr{P}_E(j\hat{\boldsymbol{j}})$, justifying the use of the expansion (D 5) and demonstrating its convergence for sufficiently small $\boldsymbol{j}$.

The author wishes to thank his research supervisor, Dr H. K. Moffatt, for his help and encouragement during this work. The Science Research Council supported the early stages with a Research Studentship at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge. It was completed at the Department of Applied Mathematics, University of Newcastle upon Tyne, while the author was working with Professor P. H. Roberts as a Senior Research Associate, with the research grant GR/3/425 from the Natural Environment Research Council.

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